linear form will be inconsistent under similar conditions. For example, the equations

$$\begin{aligned} ax^2 + by^2 + c_1 &= 0, \\ ax^2 + by^2 + c_2 &= 0, \end{aligned} \qquad c_1 \neq c_2, \tag{19}$$

are linear in x^2 and y^2. Subtracting we should get constant $= 0$ so that the equations are inconsistent in the usual sense. But using formula (15) we find $x^2 = \infty$, $y^2 = \infty$ and we may say that the equations have two pairs of equal roots, *i.e.*, the curves have double contact at infinity.

Two equations need not be of the same degree to be inconsistent. An instance of this already noticed is the equation of a hyperbola and that of its asymptote. Another example is furnished by the equations

$$\begin{aligned} y^3 - xy^2 - x^2 + 2xy &= 0, \\ y^2 - y + x + 1 &= 0, \end{aligned} \tag{20}$$

which, on elimination of x, yield

$$0y^4 + 0y^3 + 0y^2 + 0y - 1 = 0. \tag{21}$$

The equations which should have six solutions thus have no finite solutions and the corresponding curves meet wholly at infinity. We may now formulate a general criterion for inconsistent equations:

r non-homogeneous (dependent or independent) equations in n variables, $r \leqq n$, are inconsistent when and only when the loci of the equations intersect, whether in real or imaginary points, wholly at infinity.

The purpose of this discussion has been not merely to criticize the shortcomings of our current elementary textbooks in respect to imaginary and infinite elements but to illustrate how these fruitful ideas can be employed to enrich both algebra and geometry. The suggestions here embodied are consonant with the spirit that pervades modern mathematics and with sound European tradition. It is my conviction that they could be incorporated into our elementary texts without sacrificing either clearness or rigor and at no greater cost of space than is required to detail the exceptions which they eliminate.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

26. Burckhardt on Modern Teaching.

The number of times the teaching of mathematics has been reformed and the general similarity of view of the reformers are always interesting to the student of the history of the subject. Every day, in educational circles, theories are launched forth that have been common property so long that it is not to be wondered that, in certain schools, the history of education is frowned upon,

ostensibly because "we must face the future" but really because the makers of these theories fear the revelations of the past. One of our perennial discoveries is that mathematics needs to have the waste material eliminated, that the pupil should study only those parts that he will use, and that the best place for this is in some form of laboratory.

Among my autographs is a letter on this subject, written somewhat over a century ago, by Johann Karl Burckhardt.[1] Since he became (1799) a naturalized French citizen, his given name more often appears as Jean Charles. He was a member of the Académie des Sciences and of the Institut (Classe des Sciences physiques et mathématiques) and, although primarily an astronomer,[2] is known for his factor tables and for his considerable interest in pure mathematics. M. Jean-Denis Barbié Du Bocage,[3] a well-known geographer and a member of the Institut, had asked Burckhardt for a plan for the mathematical training of his son, and the result was a letter which, in view of its modern ideas, might well form an appendix to any of the recent reports on the improvement of the teaching of the subject. The following is a translation:

Monday, August 30, 1813

My celebrated Colleague,

I have thought a great deal since yesterday about the best thing to do for your son. It seems to me that there are two points which ought to be considered: First, that it is frequently the case that there are taught for long periods many things which are useless from the practical standpoint and for the purpose which I have in mind for your son, the result being that there is not left sufficient time for the necessary features. Second, that the pupil should not be discouraged in his mathematical studies by any difficulties which he may encounter.

Here, then, is the plan which seems to me the most practicable:

(1) The teacher should present only those features of plane geometry which are necessary for the complete understanding of the theorems which form the foundation of surveying. He should therefore omit such non-essentials as the actual computation of the value of the number $3.14159\cdots$, a thing for which your son would have neither the time nor the need.

(2) After this the instructor should teach algebra up to and including equations of the second degree. The first chapters of Euler[4] form a good model for this kind of work and also include the theory of logarithms. I believe that this study is necessary, if only for the purpose of knowing how to read and use those algebraic formulas which are so necessary in practical mathematics.

(3) Plane trigonometry.

(4) Spherical trigonometry, which should be preceded by certain necessary theorems of geometry which were not taken up in the earlier course. The treatment which M. Delambre has given in his *Abrégé*[5] seems to me to merit preference.

(5) I have not mentioned solid geometry, but it is easy to determine a geographic position[6] even if ignorant that the cone is a third of a cylinder, &c, &c. The teacher may therefore give whatever the time permits.

Perhaps the difficulty will lie in finding a teacher who can carry out this idea. The plan requires more work on his part, fewer [private] lessons and [hence] a diminished financial return. Perhaps, also, he may hesitate to make himself known through fear that his colleagues will decry

[1] Born at Leipzig, April 30, 1773; died at Paris, June 21, 1825.

[2] I have, for example, the original manuscripts of his papers "Sur la cométe actuelle" and "Tables de la Lune" (introduction), presented to the Institut in 1811, each with Delambre's signed memorandum.

[3] Born at Paris in 1760; died in 1825. He founded the Société de Géographie in 1821.

[4] His algebra had appeared at Petrograd (Petersburg) in 1770 and had later been translated into French and English.

[5] *Abrégé d'astronomie* (Paris, 1813) which had just appeared.

[6] Referring to M. Barbié Du Bocage's standing as a leading geographer.

him as disloyal to his profession.[1] On the other hand it is quite possible to lead the teacher to see that this work, once done, will serve to assist others, since circumstances will often force parents to demand more expeditious teaching and such as is directed towards practical ends.

With respect to the necessary equipment, it should be ordered early, since the makers are often engaged for a long time ahead.

Finally, on Nov. 8 and Nov. 9 certain stars will be in eclipse about 6 o'clock in the evening. If your son wishes to observe them with me, he may come about 5 o'clock. If the weather should be unsettled, he would not need to come, for other opportunities will offer themselves.

With sentiments of high consideration, I am,

Your servant and colleague
BURCKHARDT.

It would be interesting to know, but exceedingly wearisome to attempt to find, the number of times that these same ideas have been advanced in the hundred and ten years which have elapsed since Burckhardt "thought a great deal since yesterday" on the great problem of mathematical education. I often wonder if some of the reformers give any more time to the subject than Burckhardt did on this occasion.

27. BURCKHARDT AND THE ETERNAL PROBLEM OF PUBLICATION.

The World War has emphasized the difficulty of publishing scientific works of every kind, both in this country and abroad. The gift of the Hegeler Trust to the Mathematical Association of America is affording relief in one important line at present, the National Research Council has secured funds to enable it to make a beginning in another line, and the Carnegie Institution of Washington has lent a hand, but there still remains the great difficulty of finding some way of publishing existing manuscripts of undoubted mathematical value and of financing the preparation of others.

In view of this state of affairs it is interesting to see occasional evidences of the fact that the problem is not merely one of the present day. It was inadequately solved in earlier times by finding a patron among the old nobility and dedicating to him a work of which he could rarely comprehend even the general nature, but after the French Revolution this method passed rapidly into disuse and the favorite method came to be by resort to communications to learned societies.

Among my autographs is a letter written by Burckhardt[2] in 1814 showing the trouble he was having with his factor table, and it reads not unlike dozens of communications of the present day. The address is mostly obliterated, but it was that of a member of the Institut. The translation is as follows:

Friday, Sept. 30, 1814

Monsieur,

I propose to send, next Monday, a letter to M. Pfaff,[3] professor at Halle, for the purpose of placing in his hands a list of books which the Institut wishes to buy at the sale of the library of M. Klügel.[4] If you wish to purchase anything from this collection, I wish you would send me a note on Monday, at the Institut.

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[1] In the original, as a *gate-metier*, one who debases his profession.

[2] See page 298, footnote 1.

[3] This was Johann Friedrich Pfaff (1765–1825), who had gone to Halle four years before this.

[4] This was Georg Simon Klügel (1739–1812), whose dictionary of mathematics appeared at Leipzig 1803–1808 (3 vols. with later fourth and fifth volumes).

During the period in which we were so anxious to carry on our scientific pursuits, I worked out the factors of the third and fourth millions.[1] Madame Courcier has offered to print them if the Institut will contribute a thousand francs towards the expenses. This is only 500 francs for each million, although the second million alone has required a sacrifice of 1000 francs. I have a little repugnance at proposing this request; nevertheless I shall gain nothing except the uncomfortable labor of correcting the proofs, which probably no one after my death would ever attempt. On the other hand, I have seen in the reports of the past year that 1500 francs was given for the printing of botanical works, and yet that the mathematical section did not make any expenditure of a similar nature.

Madame Courcier has a good opinion of the sale of this work, since the English have bought several copies of the second million; if this hope is not maintained, she will probably increase her demand to such an extent that it would seem perhaps advisable to profit by the offer which she now makes.

I have the honor of being, with respect,
Monsieur,

V. T. H. & T. O. S,[2]
BURCKHARDT

Below there is a memorandum in Delambre's hand:

"Granted 1000 fr. 7 November 1814."

The letter reads so much like several that have come to my attention in recent years that I feel that it may contain some suggestions of value and some words of encouragement to those who have to meet the problem at the present time.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

NEW QUESTION.

The following question was asked by a member of the Ohio Section of the Association, at the meeting of April 20, 1922:

47. Can anyone teaching the theory of investment furnish references for bond valuation formulas when the income rate is a varying function of the time?

DISCUSSIONS.

In the discussion concerning definitions, Professor Allen draws attention to the inexcusable carelessness of many writers on algebra in the terminology of complex numbers. It seems a reasonable demand that an author should use the terms "real," "imaginary" and "complex" in such a way that it may be definitely determined whether or not a given number belongs to one of these classes; and it is rather surprising to learn how few writers of texts have taken the trouble to do this consistently.

The vague and unsettled use of the word "imaginary" in former times can scarcely be blamed for the present situation. It is true that negative numbers

[1] That is, of his *Table des diviseurs pour tous le nombres des premier, deuxième et troisième millions, avec les nombres premiers qui s'y trouvent,* which was published in complete form, at Paris, in 1817.

[2] Votre toute honoré et tout obeissant serviteur.